XXVIII

ELASTICITY SOLUTIONS

Cylindrical polar coordinates (r, θ, z) are used in the analysis. Axial symmetry is assumed; the stresses are independent of the angle θ. End effects are not considered*; the stresses found are independent of the axial coordinate z.

Elasticity Solutions for a Cylinder

The two-dimensional solutions for a cylinder loaded by uniform inner and outer pressures is given by Timoshenko and Goodier[41]. The expressions for stresses and displacement in cylinder n are

$$\sigma_r = \frac{1}{k_n^2 - 1}\left[p_{n-1} - p_n k_n^2 - (p_{n-1} - p_n)\left(\frac{r_n}{r}\right)^2\right]$$

$$\sigma_\theta = \frac{1}{k_n^2 - 1}\left[p_{n-1} - p_n k_n^2 + (p_{n-1} - p_n)\left(\frac{r_n}{r}\right)^2\right] \qquad (13a\text{-}c)$$

$$\tau_{r\theta} = 0$$

$$\frac{u}{r} = \frac{1}{E_n(k_n^2 - 1)}\left[(1-\nu)(p_{n-1} - p_n k_n^2) + (1+\nu)(p_{n-1} - p_n)\left(\frac{r_n}{r}\right)^2\right]$$

$$v = 0 \qquad (14a, b)$$

where σ_r, σ_θ, and $\tau_{r\theta}$ are the radial stress, hoop stress, and shear stress, respectively, and where u and v are the radial and circumferential displacements, respectively. (The radii r_n, the pressures p_n, and the wall ratios k_n have been defined previously.) Equation (13a-c) also gives the residual stresses if the operating pressures p_n are replaced by the residual pressures q_n.

For a fatigue analysis of a cylinder of ductile material the range and mean shear stresses are needed. The greatest range in the shear stress in a cylinder occurs at the bore on a plane oriented at 45 degrees to the r and θ axes. The shear stress there is given by

$$S = \frac{\sigma_\theta - \sigma_r}{2} \qquad (15)$$

*It may be important to consider end effects depending upon the method of end closure in the design. These effects and possible axial stresses resulting from large shrink fits may not be negligible.

Formulating the range in stress from the Definition (6a), we get

$$S_r = \frac{1}{2}\left[\frac{\sigma_\theta(p_n, p_{n-1}) - \sigma_r(p_n, p_{n-1})}{2} - \frac{\sigma_\theta(q_n, q_{n-1}) - \sigma_r(q_n, q_{n-1})}{2}\right] \text{ at } r = r_{n-1} \quad ,$$

hence,

$$S_r = \frac{k_n^2}{2(k_n^2-1)}\left[(p_{n-1} - p_n) - (q_{n-1} - q_n)\right], \text{ at } r = r_{n-1} \tag{16}$$

The mean shear stress at the same location on the same plane is

$$S_m = \frac{k_n^2}{2(k_n^2-1)}\left[(p_{n-1} - p_n) + (q_{n-1} - q_n)\right], \text{ at } r = r_{n-1} \tag{17}$$

Elasticity Solutions for Segmented Components

Elasticity solutions for the segments were derived. The derivations are outlined in Appendix I and only the results are given here. There are two types of segments. The ring segment is loaded by p_1 at r_1 and by p_2 at r_2. The pin segment is loaded by p_1 at r_1 but by more complex loading at r_2.

Ring Segment

The results for the ring segment are:

$$\sigma_r = (\sigma_r)_c + \frac{4M_1p_1}{\beta_1} f_1(r)$$

$$\sigma_\theta = (\sigma_\theta)_c + \frac{4M_1p_1}{\beta_1} f_2(r) \tag{18a-c}$$

$$\tau_{r\theta} = 0$$

$$\frac{u}{r} = (u)_c + \frac{M_1p_1}{E_2\beta_1} f_3(r) + \frac{G_1p_1}{r} \cos\theta$$

$$\frac{v}{r} = \frac{8M_1p_1}{E_2\beta_1}(k_2^2-1)\,\theta - \frac{G_1p_1}{r} \sin\theta \tag{19a, b}$$